

# THE SPINE OF A FOURIER-STIELTJES ALGEBRA: CORRIGENDA

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It has come to the authors' attention that there are several errors in our paper [3]. Fortunately, these errors are correctable. Some of these errors carry on to modest, though mainly cosmetic, errors in follow-up papers [5] and [4].

We are grateful to Elçim Elgun for pointing out the gap in the proof of [3, Theorem 2.2] and the error in the proof of [3, Theorem 5.1]. We are also grateful to Pekka Salmi for pointing out the flaw in the statement of [3, Theorem 4.2] and suggesting its correct form.

We appeal to [3] for pertinent notation and terminology.

## 1. ON NON-QUOTIENT LOCALLY PRECOMPACT TOPOLOGIES

The following should replace [3, Theorem 2.2].

**Theorem 1.1.** *Let  $\tau \in \mathcal{T}(G)$ . Then the following hold.*

- (i) *There exists a unique  $\tau_{nq}$  in  $\mathcal{T}_{nq}(G)$  such that  $\tau$  is a quotient of  $\tau_{nq}$ .*
- (ii) *If  $G$  is abelian, then  $\tau_{nq} = \tau \vee \tau_{ap}$ .*

In [4, Theorem 2.2] it is claimed that  $\tau_{nq} = \tau \vee \tau_{ap}$  for general locally compact groups  $G$  and  $\tau$  in  $\mathcal{T}(G)$ . This is false. For example let  $G = \mathrm{SL}_2(\mathbb{R})$ . We have that, on  $G$ ,  $\tau_{ap}$  is the trivial topology  $\varepsilon = \{\emptyset, G\}$ . Now if  $q : G \rightarrow G/\{-I, I\}$  is the quotient map and  $\tau = q^{-1}(\tau_{G/\{-I, I\}})$  is the coarsest topology making  $q$  continuous, then we have that  $\tau_{nq} = \tau_G$  whereas  $\tau \vee \tau_{ap} = \tau \subsetneq \tau_G$ .

The proof of part (ii) proceeds exactly as does the proof of [3, Theorem 2.2]. In particular, in the second paragraph of that proof, [3, Lemma 2.3] may be used to show that  $s \mapsto (\eta_\tau^{\tau_1}(s), \eta_{ap}^{\tau_1}) : G_{\tau_1} \rightarrow G_\tau \times G_{\tau_1}^{ap}$  is a bicontinuous isomorphism, since  $G_{\tau_1}$ , being abelian, is maximally almost periodic.

The proof of part (i), however, demands more care. We fix  $\tau_0$  in  $\mathcal{T}(G)$  and let

$$(1.1) \quad \mathcal{Q}_{\tau_0} = \{\tau \in \mathcal{T}(G) : \tau_0 \text{ is a quotient of } \tau\}.$$

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**Lemma 1.2.** (i) If  $\tau_1, \tau_2 \in \mathcal{Q}_{\tau_0}$  then  $\tau_1 \vee \tau_2 \in \mathcal{Q}_{\tau_0}$  as well.

(ii) If  $\tau_1, \tau_2 \in \mathcal{T}(G)$  satisfy  $\tau_0 \subseteq \tau_1 \subseteq \tau_2$  and  $\tau_2 \in \mathcal{Q}_{\tau_0}$ , then  $\tau_1$  is a quotient of  $\tau_2$ .

**Proof.** (i) For  $j = 1, 2$  we let  $K_0^j = \ker \eta_{\tau_0}^{\tau_j}$ . Our assumptions provide that for  $j = 1, 2$ ,  $K_0^j$  is compact and  $G_{\tau_j}/K_0^j \cong G_{\tau_0}$ . We identify  $G_{\tau_1 \vee \tau_2}$  as a subgroup of  $G_{\tau_1} \times G_{\tau_2}$  and define  $K = G_{\tau_1 \vee \tau_2} \cap (K_0^1 \times K_0^2)$ . We let  $q : G_{\tau_1 \vee \tau_2} \rightarrow G_{\tau_1 \vee \tau_2}/K$  denote the quotient map. We obtain the following commutative diagram

$$\begin{array}{ccc} G_{\tau_1 \vee \tau_2} & \xrightarrow{\quad} & G_{\tau_1} \times G_{\tau_2} \\ \downarrow q & & \searrow \eta_{\tau_0}^{\tau_1} \times \eta_{\tau_0}^{\tau_2} \\ G_{\tau_1 \vee \tau_2}/K & \xrightarrow{\quad} & G_{\tau_1} \times G_{\tau_2}/(K_0^1 \times K_0^2) \cong G_{\tau_1}/K_0^1 \times G_{\tau_2}/K_0^2 \xrightarrow{\sim} G_{\tau_0} \times G_{\tau_0}. \end{array}$$

Identifying  $G_{\tau_0}$  with the diagonal subgroup of  $G_{\tau_0} \times G_{\tau_0}$ , this diagram shows that the map  $\eta_{\tau_0}^{\tau_1 \vee \tau_2} = \eta_{\tau_0}^{\tau_1} \times \eta_{\tau_0}^{\tau_2}|_{G_{\tau_1 \vee \tau_2}}$  is a proper map.

(ii) We recall that our assumptions give the following commuting diagram

$$(1.2) \quad \begin{array}{ccc} G_{\tau_2} & \xrightarrow{\eta_{\tau_1}^{\tau_2}} & G_{\tau_1} \\ & \searrow \eta_{\tau_0}^{\tau_2} & \downarrow \eta_{\tau_0}^{\tau_1} \\ & & G_{\tau_2}/K_0^2 \xrightarrow{\sim} G_{\tau_0} \end{array}$$

where  $K_0^2 = \ker \eta_{\tau_0}^{\tau_2}$  is compact. Then  $K_1^2 = \ker \eta_{\tau_1}^{\tau_2} \subset K_0^2$ , and is thus compact. We let  $q_1^2 : G_{\tau_2} \rightarrow G_{\tau_2}/K_1^2$  be the quotient map and  $\tau = (q_1^2 \circ \eta_{\tau_2})^{-1}(\tau_{G_{\tau_2}/K_1^2})$ , so that  $G_{\tau} \cong G_{\tau_2}/K_1^2$ ,  $\tau \supseteq \tau_1$  and  $\eta_{\tau_1}^{\tau}$  is injective. The commuting diagram (1.2) and the first isomorphism theorem give the commuting diagram

$$(1.3) \quad \begin{array}{ccccc} G_{\tau_2} & \xrightarrow{\eta_{\tau}^{\tau_2}} & G_{\tau} & \xrightarrow{\eta_{\tau_1}^{\tau}} & G_{\tau_1} \\ & \searrow \eta_{\tau_0}^{\tau_2} & \searrow q & & \downarrow \eta_{\tau_0}^{\tau_1} \\ & & G_{\tau}/K & \xrightarrow{\sim} & G_{\tau_0} \end{array}$$

where  $K = K_0^2/K_1^2$  and  $q : G_{\tau} \rightarrow G_{\tau}/K$  is the quotient map.

Since  $\eta_{\tau_1}^{\tau}$  is injective, it suffices to prove that it is open for  $\eta_{\tau_1}^{\tau_2}$  to be a quotient map, i.e. we obtain that  $\tau_1 = \tau$  where  $\eta_{\tau}^{\tau_2}$  is a quotient map. To this end, let  $U \subset G_{\tau}$  be relatively compact open set. Then  $UK$  is also relatively compact and open. Hence  $\eta_{\tau_1}^{\tau}(\overline{UK})$  is closed and equal to  $\eta_{\tau_1}^{\tau}(UK)$ , and furthermore

$$(1.4) \quad \eta_{\tau_1}^{\tau}|_{\overline{UK}} : \overline{UK} \rightarrow \overline{\eta_{\tau_1}^{\tau}(UK)} \text{ is a homeomorphism.}$$

The commuting diagram (1.3) tells us that

$$\eta_{\tau_1}^{\tau}(UK) = (\eta_{\tau_0}^{\tau_1})^{-1}(q(U))$$

which is open in  $G_{\tau_1}$ . Hence by (1.4),

$$\eta_{\tau_1}^\tau|_{UK} : UK \rightarrow \eta_{\tau_1}^\tau(UK)$$

is a homeomorphism onto an open subset so  $\eta_{\tau_1}^\tau(U)$  is open.  $\square$

**Proof of Theorem 1.1 (i).** We first note that  $\mathcal{Q}_\tau$ , as defined in (1.1), is a directed system: if  $\tau_1, \tau_2 \in \mathcal{Q}_\tau$  then  $\tau_1 \vee \tau_2 \in \mathcal{Q}_\tau$  by Lemma 1.2 (i). Moreover, it follows Lemma 1.2 (ii) that if  $\tau_1, \tau_2 \in \mathcal{Q}_\tau$  with  $\tau_1 \subseteq \tau_2$ , then  $\eta_{\tau_1}^{\tau_2}$  is a proper map. Hence the inverse mapping system

$$\{G_{\tau'}, \eta_{\tau_1}^{\tau_2} : \tau' \in \mathcal{Q}_\tau, \tau_1 \subset \tau_2 \text{ in } \mathcal{Q}_\tau\}$$

gives rise to the projective limit

$$\begin{aligned} G_{\mathcal{Q}_\tau} &= \varprojlim_{\tau' \in \mathcal{Q}_\tau} G_{\tau'} = \left\{ (s_{\tau'}) \in \prod_{\tau' \in \mathcal{Q}_\tau} G_{\tau'} : \eta_{\tau_1}^{\tau_2}(s_{\tau_2}) = s_{\tau_1} \text{ if } \tau_1 \subseteq \tau_2 \text{ in } \mathcal{Q}_\tau \right\} \\ &= \left\{ (s_{\tau'}) \in \prod_{\tau' \in \mathcal{Q}_\tau} G_{\tau'} : \eta_{\tau'}^{\tau'}(s_{\tau'}) = s_\tau \text{ for } \tau' \text{ in } \mathcal{Q}_\tau \right\} \end{aligned}$$

which is locally compact by [3, Proposition 2.1]. We let  $\tau_{nq}$  denote the coarsest topology which makes the map  $s \mapsto (\eta_{\tau'}(s)) : G \rightarrow G_{\mathcal{Q}_\tau}$ , continuous. We have that  $\tau_{nq} \supseteq \tau'$  for every  $\tau'$  in  $\mathcal{Q}_\tau$ .

We now show that  $\tau_{nq} \in \mathcal{Q}_\tau$ . First observe that  $\eta_\tau^{\tau_{nq}} : G_{\tau_{nq}} \cong G_{\mathcal{Q}_\tau} \rightarrow G_\tau$  is given by the map  $(s_{\tau'}) \mapsto s_\tau$ . With this identification we have that

$$\ker \eta_\tau^{\tau_{nq}} = \varprojlim_{\tau' \in \mathcal{Q}_\tau} \ker \eta_\tau^{\tau'}$$

and is thus compact. Moreover  $\eta_\tau^{\tau_{nq}}$  is open, since for any basic open set

$$V = \prod_{\tau' \in \mathcal{Q}_\tau \setminus \{\tau'_1, \dots, \tau'_n\}} G_{\tau'} \times \prod_{j=1}^n U_{\tau'_j} \subset \prod_{\tau' \in \mathcal{Q}_\tau} G_{\tau'}$$

where each  $U_{\tau'_j}$  is open in  $G_{\tau'_j}$ , we have

$$\eta_\tau^{\tau_{nq}}(V \cap G_{\mathcal{Q}_\tau}) = \bigcap_{j=1}^n \eta_{\tau'}^{\tau'_j}(U_{\tau'_j})$$

where each  $\eta_{\tau'}^{\tau'_j}(U_{\tau'_j})$  is open by assumption.

Finally, if there were  $\tau_1$  in  $\mathcal{T}(G)$  of which  $\tau_{nq}$  is a quotient, then by the first isomorphism theorem we would have that  $\tau_1 \in \mathcal{Q}_\tau$ . Hence  $\tau_1 \subseteq \tau_{nq}$ . Thus  $\tau_{nq}$  is a non-quotient topology, and the unique such one of which  $\tau$  is a quotient.  $\square$

For any locally compact group  $G$  for which  $\tau_{nq} = \tau \vee \tau_{ap}$  for any  $\tau$  in  $\mathcal{T}(G)$ , we obtained in [3, Section 2.4] that  $\mathcal{T}_{nq}(G) = \mathcal{T}(G) \vee \tau_{ap}$  and is thus an ideal in, and hence a subsemilattice of, the semilattice  $(\mathcal{T}(G), \vee)$ . We note that for any  $\tau$  for which  $G_\tau$  is maximally almost periodic, we have  $\tau_{nq} = \tau \vee \tau_{ap}$ .

Unfortunately, it is not clear whether  $\mathcal{T}_{nq}(G)$  is a subsemilattice of  $\mathcal{T}(G)$ , in general. However the following is immediate.

**Corollary 1.3.** *If  $\tau_1, \tau_2 \in \mathcal{T}_{nq}(G)$  define*

$$\tau_1 \tilde{\vee} \tau_2 = (\tau_1 \vee \tau_2)_{nq}.$$

*Then  $(\mathcal{T}_{nq}(G), \tilde{\vee})$  is a quotient semilattice of  $(\mathcal{T}(G), \vee)$ .*

The only inobvious aspect of this corollary is the associativity of  $\tilde{\vee}$ . We observe that  $\tau_1 \vee (\tau_2 \tilde{\vee} \tau_3)$  admits  $\tau_1 \vee \tau_2 \vee \tau_3$  as a quotient, and hence, by Lemma 1.2 (ii), is itself a quotient of  $(\tau_1 \vee \tau_2 \vee \tau_3)_{nq}$ . Symmetrically, the same is true of  $(\tau_1 \tilde{\vee} \tau_2) \vee \tau_3$ . Hence

$$\tau_1 \tilde{\vee} (\tau_2 \tilde{\vee} \tau_3) = (\tau_1 \vee \tau_2 \vee \tau_3)_{nq} = (\tau_1 \tilde{\vee} \tau_2) \tilde{\vee} \tau_3.$$

Unless it can be shown that  $\tau_1 \tilde{\vee} \tau_2 = \tau_1 \vee \tau_2$  for all  $\tau_1, \tau_2 \in \mathcal{T}_{nq}(G)$ , some changes have to be made to the exposition in [3, Section 4], where  $\vee$  must always be replaced by, or understood to be,  $\tilde{\vee}$ . Fortunately, this change does not appear to affect any of the results or proofs in this section, or any later part of the paper, in more than a cosmetic manner.

## 2. TOPOLOGY OF THE SPINE COMPACTIFICATION

As shown in [3, Theorem 4.1], with appropriate notational changes as suggested by Corollary 1.3, the spectrum of the algebra  $A^*(G)$  is given by

$$G^* = \bigsqcup_{\mathcal{S} \in \mathfrak{H}\mathfrak{D}(G)} G_{\mathcal{S}}$$

where each  $G_{\mathcal{S}}$  is a projective limit over a hereditary directed subset  $\mathcal{S}$  of  $(\mathcal{T}_{nq}(G), \tilde{\vee})$ . The statement of [3, Theorem 4.2] is flawed, and should be replaced with the following.

**Theorem 2.1.** *The topology on  $G^*$  is given as follows: for any  $s_0$  in  $G^*$ , say  $s_0 \in G_{\mathcal{S}_0}$  for some  $\mathcal{S}_0$  in  $\mathfrak{H}\mathfrak{D}(G)$ , a neighbourhood base at  $s_0$  is formed by the sets*

$$\begin{aligned} U(V_{\tau}; W_{\tau_1}, \dots, W_{\tau_n}) = \{ s \in G^* : s \in G_{\mathcal{S}} \text{ for some } \mathcal{S} \supseteq \mathcal{S}_{\tau} \text{ in } \mathfrak{H}\mathfrak{D}(G) \\ \text{with } s_{\tau} \in V_{\tau}, \text{ and } s_{\tau_j} \in W_{\tau_j} \text{ if} \\ \mathcal{S} \supseteq \mathcal{S}_{\tau_j}, \text{ for } j = 1, \dots, n \} \end{aligned}$$

where  $\tau \in \mathcal{S}_0$ ,  $\tau_1, \dots, \tau_n \in \mathcal{T}_{nq}(G) \setminus \mathcal{S}_0$ ,  $V_{\tau}$  is an open neighbourhood of  $s_{0,\tau}$  in  $G_{\tau}$ , and each  $W_{\tau_j}$  is a cocompact subset of  $G_{\tau_j}$ .

Pekka Salmi has pointed out to us that the error in the description of [3, Theorem 4.2], implies that all groups  $G_{\mathcal{S}}$ , for  $\mathcal{S} \in \mathfrak{H}\mathfrak{D}(G)$  are locally compact. This is false, as is implicit in [3, Section 6.3], or is shown in [1, Theorem 2].

Unfortunately, the proof of [3, Theorem 4.2] requires slight modification.

**Proof of Theorem 2.1.** We should first observe that the family of sets described above indeed is a base for a topology. It is straightforward to check that for  $\tau, \tau' \in \mathcal{S}_0$  and  $\tau_1, \dots, \tau_n, \tau'_1, \dots, \tau'_m$  in  $\mathcal{T}_{nq}(G) \setminus \mathcal{S}_0$  that

$$\begin{aligned} & U(V_\tau; W_{\tau_1}, \dots, W_{\tau_n}) \cap U(V_{\tau'}; W_{\tau'_1}, \dots, W_{\tau'_m}) \\ & \supset U\left((\eta_\tau^{\tau \vee \tau'})^{-1}(V_\tau) \cap (\eta_{\tau'}^{\tau \vee \tau'})^{-1}(V_{\tau'}); W_{\tau_1}, \dots, W_{\tau_n}, W_{\tau'_1}, \dots, W_{\tau'_m}\right) \end{aligned}$$

for neighbourhoods  $V_\tau$  of  $s_{0,\tau}$ ,  $V_{\tau'}$  of  $s_{0,\tau'}$ , and cocompact subsets  $W_{\tau_1}, \dots, W_{\tau'_m}$  of  $G_{\tau_1}, \dots, G_{\tau'_m}$ , respectively.

We note that for a net  $(s_i)_{i \in I}$  in  $G^*$ , the following are equivalent:

- (i)  $s_i \rightarrow s_0$  in  $G^*$  with the topology described above;
- (ii) for each  $\tau_0 \in \mathcal{S}_0$  there is  $i_0$  such that  $i \geq i_0$  implies that  $s_i \in G_{\mathcal{S}_i}$  for some  $\mathcal{S}_i \supset \mathcal{S}_{\tau_0}$  and  $\lim_{i \geq i_0} s_{i,\tau} = s_{0,\tau}$ ; and for each  $\tau \notin \mathcal{S}_0$  for which  $I_\tau = \{i : s_i \in \mathcal{S}_i \text{ for some } \mathcal{S}_i \supset \mathcal{S}_\tau\}$  admits no maximal element in  $I$ , and for any co-compact  $W_\tau \subset G_\tau$ , there is  $i_\tau$  in  $I$  for which  $s_{i,\tau} \in W_\tau$  if  $i \in I_\tau$  and  $i \geq i_\tau$ ;

- (iii)  $\chi_{s_i} \rightarrow \chi_{s_0}$  weak\* in  $A^*(G)^*$ .

The equivalence of (i) and (ii) is clear. If we write  $u$  in  $A^*(G)$  as  $u = \sum_{\tau \in \mathcal{T}_{nq}(G)} u_\tau$  as in [3, (4.2)], and suppose (ii), above, then

$$\chi_{s_i}(u) = \sum_{\tau \in \mathcal{S}_0 \setminus \mathcal{S}_i} \hat{u}_\tau(s_{i,\tau}) + \sum_{\tau \in \mathcal{S}_i \setminus \mathcal{S}_0} \hat{u}_\tau(s_{i,\tau}) \rightarrow \sum_{\tau \in \mathcal{S}_0} \hat{u}_\tau(s_{0,\tau}) = \chi_{s_0}(u)$$

where  $s_i \in \mathcal{S}_i$  for each  $i$ ; which shows (iii). Likewise, selecting  $u = u_\tau$ , a repeat of the computation above shows that (iii) implies (ii).  $\square$

We remark that [3, Corollary 4.3] remains unchanged. The neighbourhood in the proof of part (ii), therein, should be changed to

$$U(V_{\tau_0}) = \{s \in G^* : s \in G_{\mathcal{S}'}, \text{ for some } \mathcal{S}' \supset \mathcal{S}_{\tau_0} \text{ and } S_{\tau_0} \in V_{\tau_0}\}.$$

Furthermore, a modest change must be made in the proof of [3, Proposition 4.6 (ii)]. If a net  $(e_i)$  of idempotents converges to  $s$  in  $G_{\mathcal{S}}$ , then for all  $\tau \in \mathcal{S}$ , there is an  $i_\tau$  for which  $\mathcal{S}_i \supset \mathcal{S}_\tau$ , where  $e_i \in G_{\mathcal{S}_i}$ , for  $i \geq i_\tau$ , and  $\lim_{i \geq i_\tau} e_{i,\tau} = s_\tau$ ; and for  $\tau$  in  $\mathcal{T}_{nq}(G) \setminus \mathcal{S}$ , there is  $i_\tau$  for which  $\mathcal{S}_i \not\supset \mathcal{S}_\tau$  for  $i \geq i_\tau$ , which may be seen by observing that otherwise such  $e_{i,\tau}$  must be within the cocompact set  $W_\tau = G_\tau \setminus \{\eta_\tau(e)\}$ .

### 3. ON ABELIAN GROUPS

The fourth paragraph of the proof of [3, Theorem 5.1] contains an error in its claim that the map from  $\widehat{G}_\tau$  to the diagonal subgroup of  $\widehat{G}_\tau \times \widehat{G}_d$  is bicontinuous is false, for obviously only its inverse is continuous. Fortunately the claim which that paragraph is attempting to establish, namely that *if  $\tau$  in  $\mathcal{T}(G)$  is such that  $\widehat{G}_\tau$  is the group  $\widehat{G}$  but with a finer locally compact group topology  $\hat{\tau}$ , then  $\tau \in \mathcal{T}_{nq}(G)$* , remains true. Recall that for any  $\tau'$  in  $\mathcal{T}(G)$ , the dual map  $\widehat{\eta_{\tau'}} : \widehat{G_{\tau'}} \rightarrow \widehat{G}$  is continuous and injective. The assumptions on  $\tau$  above imply that  $\widehat{\eta_\tau}$  is also surjective. Since the map

$\eta_\tau^{\tau_{nq}} : G_{\tau_{nq}} \rightarrow G_\tau$  is proper, its dual map  $\widehat{\eta_\tau^{\tau_{nq}}}$  is open; see [2, (23.24)(d)], for example. Since  $\eta_\tau = \eta_\tau^{\tau_{nq}} \circ \eta_{\tau_{nq}}$  we have  $\widehat{\eta_\tau} = \widehat{\eta_\tau^{\tau_{nq}}} \circ \widehat{\eta_{\tau_{nq}}}$ . Thus it follows that  $\widehat{\eta_\tau^{\tau_{nq}}}$  is continuous, bijective and open, hence  $\widehat{\tau} = \widehat{\tau_{nq}}$ . Thus by Pontryagin duality  $\tau = \tau_{nq}$ .

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